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BENDING, VIBRATION AND BUCKLING OF SIMPLY SUPPORTED THICK ORTHOTROPIC RECTANGULAR PLATES AND LAMINATES

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Abstract—This paper presents a unified exact analysis for the statics and dynamics of a class of thick laminates. A three-dimensional, linear, small deformation theory of elasticity solution is developed for the bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates. All the nine elastic constants of orthotropy are taken into account. The solution is formally exact and leads to simple infinite series for stresses and displacements in flexure, forced vibration and "beam–column" type problems and to closed form characteristic equations for free vibration and buckling problems. For free vibration of plates, the present analysis yields a triply infinite spectrum of frequencies instead of only one doubly infinite spectrum by thin plate theory or three doubly infinite spectra by Reissner–Mindlin type analyses. Some numerical results are presented for plates and laminates. Comparison of results from thin plate, Reissner and Mindlin analyses with these yield some important conclusions regarding the validity and effects of the assumptions made in the approximate theories.

NOTATION

a, b, h	length, width and total thickness
E	an elastic modulus used in defining λ
k_x	buckling stress parameter $= \dfrac{12}{\pi^2}\left(\dfrac{b}{h}\right)^2 \dfrac{P_{x_{crm}}}{E_x}$
L	differential operator $\dfrac{d}{dZ}$
M, N	$m\pi h/a$, $n\pi h/b$
P_x, P_y	normal stresses on the edges $x = 0, a$ and $y = 0, b$
$P_{x_{crm}}$	critical buckling stress when plate is loaded on edges $x = 0, a$ only
X, Y, Z	x/a, y/b, z/h
λ	eigenvalue $= \sqrt{\Omega^2\rho h^2/E}$ in vibration problems
	$= \sqrt{(P_xM^2 + P_yN^2)/E}$ in problems with normal edge loads
ρ	mass density
Ω	angular frequencies—forced or free
% error	$\left\{\dfrac{\text{thin plate theory value}}{\text{value by present exact analysis}} - 1\right\} \times 100$

For laminated plates

Subscript j	($j = 1, 2, \ldots, p$) denotes corresponding ply
H_j	$\left(\sum_{i=1}^{j} h_i\right) \Big/ h$
p	number of plies
β	modular ratio between top and middle plies of a three-ply laminate (with identical top and bottom plies and also identical relative moduli within each ply)
	$= E_{x_1}/E_{x_2} = E_{x_3}/E_{x_2}$

Figure 1 indicates the coordinate system and the dimensions

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Orthotropic stress–strain relations [11]:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} F_x & F_{xy} & F_{xz} & 0 & 0 & 0 \\ F_{xy} & F_y & F_{yz} & 0 & 0 & 0 \\ F_{xz} & F_{yz} & F_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{xy} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{xz} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{yz} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

or

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} E_x & E_{xy} & E_{xz} & 0 & 0 & 0 \\ E_{xy} & E_y & E_{yz} & 0 & 0 & 0 \\ E_{xz} & E_{yz} & E_z & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{xy} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{xz} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{yz} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

INTRODUCTION

THIN plate theory is based on several assumptions, the most important of which are the neglect of transverse shear deformations and rotatory inertia. The errors in such a theory naturally increase as the plate thickness increases. Further, due to the neglect of transverse shear deformations and σ_z, one cannot take into account all the 9 elastic flexibility con-

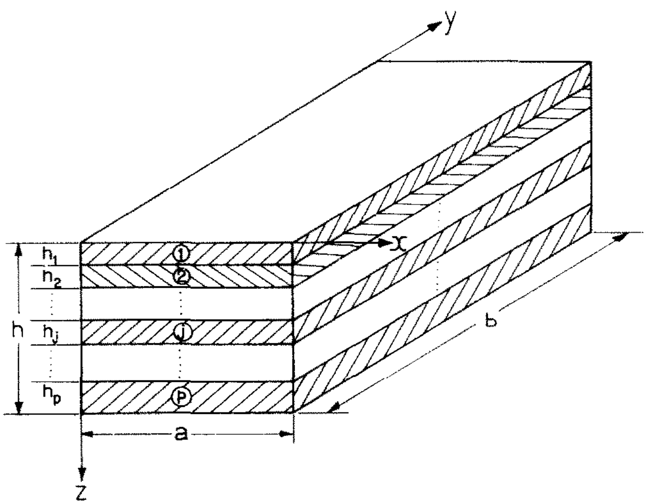

FIG. 1. Coordinate system and dimensions.

stants of orthotropy. Consequently, the errors increase as the magnitudes of the transverse elastic constants (F_z, F_{xz}, F_{yz}, $1/G_{xz}$, $1/G_{yz}$) increase relative to the in-plane constants (F_x, F_y, F_{xy}, $1/G_{xy}$). So, it is natural to seek some improved formulations which account, at least approximately, for transverse shear deformations and rotatory inertia. Reissner [1] and Mindlin [2] initiated such analyses by their modified plate theories for flexure and vibration of isotropic thick plates. They start with the standard thin plate assumptions for variation of stresses (or displacements) across thickness, but include the *resulting* transverse shear deformations and rotatory inertia. Both these analyses permit satisfaction of three boundary conditions on each edge, but they do not satisfy the governing differential equations of three-dimensional elasticity exactly. Medwadowski [3] extended Reissner's theory to orthotropic plates. Yang *et al.* [4] have developed a Mindlin-type analysis for heterogeneous plates or laminates with general anisotropy and elastic moduli varying continuously across each ply. They have also presented numerical results for plane waves. Whitney [5] has provided another analysis for anisotropic laminates taking into account transverse shear deformations; he does not presume linear variation of displacements u and v across thickness. Several other papers relevant to anisotropic laminates are published in Ref. [6].

The next obvious step is to seek an exact analysis by three-dimensional theory of elasticity. This was recently achieved for certain plates and laminates of isotropic materials [7–10]. The current interest in thick laminates made up of orthotropic layers, indicates that a three-dimensional analysis for the statics and dynamics of thick laminates made up of general orthotropic plies would be useful. In this paper such an analysis is developed for simply supported rectangular plates and laminates. Flexure, forced and free vibrations, and a class of buckling and beam-column type of problems, are all treated together.

In the analysis herein, the boundary conditions for a simple support on a straight edge, x = const., are specified as,

$$w = 0, \quad v = 0 \quad \text{and} \quad \sigma_x = 0 \text{ for all } z \tag{1}$$

The combination of edge conditions in equation (1) amounts to providing an edge support infinitely rigid in its own plane ($w = 0$, $v = 0$ on x = const.), but completely flexible to out of plane stresses ($\sigma_x = 0$, for x = const.). In case of plates under normal edge loads, equation (1) refers to the quantities arising out of deviation from the original state of uniform strain.

For a homogeneous plate, the solution is set up in the form of a double trigonometric series in Cartesian coordinates, [*vide* equation (4) to follow], for each of the displacements u, v, w such that the governing differential equations and all the edge conditions are identically satisfied. The terms corresponding to each harmonic in the series contain six arbitrary constants and these are explicitly obtained by solving six simultaneous equations, which result from satisfying the appropriate lateral surface conditions.

The analysis for laminates is a direct extension of that for a homogeneous plate; each ply is treated as an individual homogeneous plate and, at the interface between any pair of plies, a set of six homogeneous conditions representing equilibrium and continuity must be satisfied. In a p-plied laminate, there are $6p$ arbitrary constants corresponding to each harmonic, i.e. each combination of m and n, and these are explicitly derived by satisfying all the relevant lateral surface conditions.

A special feature of this analysis is that all homogeneous surface conditions (which include the interface conditions in laminates) are satisfied identically.

In the eigenvalue problems of free vibration or buckling, a closed form characteristic equation (in determinant form, of order $6p$) is obtained for each combination of m and n. Each such equation yields an infinite number of eigenvalues, each one of them representing a different "thickness mode". The first of these is generally referred to as a "flexural mode" and it is only this mode which thin plate theory can identify and yield approximate results for. In many buckling problems one of the flexural modes is the primary mode of instability.

In thin plate theory, only $w(x, y)$ is given freedom. In Mindlin's, $\partial u/\partial z(x, y)$ and $\partial v/\partial z(x, y)$ and in Reissner's, $M_x(x, y)$ and $M_y(x, y)$ are also given freedom. In the present exact analysis u, v, w at every point are given freedom. Correspondingly for free vibration, thin plate theory, Reissner–Mindlin analyses and the exact solution respectively yield one, three and an infinite number doubly infinite sets of eigenvalues.

As the number of variables in orthotropic plates or laminates is large, detailed study of the effects of individual variables on the physical aspects of the problems, or on the errors due to thin plate assumptions, are not undertaken; but some broad conclusions are drawn from the numerical results for an example of orthotropy. The properties of this material are listed in Table 1. Some numerical results are also presented for three-ply laminates with identical top and bottom plies; the relative values of the moduli are the same in all the plies, i.e. $(E_x : E_y : E_z : E_{xy} : E_{xz} : E_{yz} : G_{xy} : G_{xz} : G_{yz})$ are identical.

TABLE 1. ORTHOTROPIC PROPERTIES ASSUMED IN EXAMPLES

$E_y/E_x = 0{\cdot}543103$	$E_z/E_x = 0{\cdot}530172$
$E_{xy}/E_x = 0{\cdot}23319$	$E_{xz}/E_x = 0{\cdot}010776$
$E_{yz}/E_x = 0{\cdot}098276$	$G_{xy}/E_x = 0{\cdot}262931$
$G_{xz}/E_x = 0{\cdot}159914$	$G_{yz}/E_x = 0{\cdot}26681$

These properties correspond to Aragonite crystals [11].

GOVERNING EQUATIONS OF THREE-DIMENSIONAL ELASTICITY

The equations of equilibrium in terms of displacements for a homogeneous orthotropic plate can be written in Cartesian co-ordinates as

$$\begin{bmatrix} E_x\dfrac{\partial^2}{\partial x^2}+G_{xy}\dfrac{\partial^2}{\partial y^2}+G_{xz}\dfrac{\partial^2}{\partial z^2} & (E_{xy}+G_{xy})\dfrac{\partial^2}{\partial x\,\partial y} & (E_{xz}+G_{xz})\dfrac{\partial^2}{\partial x\,\partial z} \\ (E_{xy}+G_{xy})\dfrac{\partial^2}{\partial x\,\partial y} & G_{xy}\dfrac{\partial^2}{\partial x^2}+E_y\dfrac{\partial^2}{\partial y^2}+G_{yz}\dfrac{\partial^2}{\partial z^2} & (E_{yz}+G_{yz})\dfrac{\partial^2}{\partial y\,\partial z} \\ (E_{xz}+G_{xz})\dfrac{\partial^2}{\partial x\,\partial z} & (E_{yz}+G_{yz})\dfrac{\partial^2}{\partial y\,\partial z} & G_{xz}\dfrac{\partial^2}{\partial x^2}+G_{yz}\dfrac{\partial^2}{\partial y^2}+E_z\dfrac{\partial^2}{\partial z^2} \end{bmatrix}$$

$$\times\begin{bmatrix} u \\ v \\ w \end{bmatrix}+\{f\}=\{0\} \tag{2}$$

where

$\{0\}$ is a (3×1) null matrix, and $\{f\}$ is a (3×1) matrix.

Further,

(a) $\{f\} = \{0\}$ with no body forces;
(b) $\{f\} = -\rho\{\ddot{u}\,\ddot{v}\,\ddot{w}\}$ for motion with no body forces other than inertia forces, $(-\rho\{\ddot{u}\,\ddot{v}\,\ddot{w}\} = \rho\Omega^2\{u\,v\,w\}$, for simple harmonic oscillations with frequency $\Omega)$;
(c) $\{f\} = F\{u\,v\,w\}$, for general applied stresses on edges; F is a function of local stresses in the initial state and second order differential operators. [$F = -(P_x\,\partial^2/\partial x^2 + P_y\,\partial^2/\partial y^2)$ for an initial state of uniform strains $\varepsilon_x, \varepsilon_y$ and $\sigma_z = 0$. In homogeneous plates this corresponds to uniform applied normal edge stresses P_x on $x = 0, a$ and P_y on $y = 0, b$].

Conditions (a), (b) and (c) can be applied together.

SIMPLY SUPPORTED RECTANGULAR PLATES

The edge boundary conditions for a simply supported rectangular plate may be specified as:

$$\begin{aligned} &\text{On } x = 0 \text{ and } a;\quad \sigma_x = 0,\quad w = 0 \quad\text{and}\quad v = 0,\\ &\text{On } y = 0 \text{ and } b;\quad \sigma_y = 0,\quad w = 0 \quad\text{and}\quad u = 0. \end{aligned} \tag{3}$$

This set of edge conditions is identically satisfied by choosing u, v, w in the following form;

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = h \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{bmatrix} \phi(Z)\cos m\pi X \sin n\pi Y \\ \psi(Z)\sin m\pi X \cos n\pi Y \\ \chi(Z)\sin m\pi X \sin n\pi Y \end{bmatrix} \tag{4}$$

where ϕ, ψ and χ are pure functions of Z. Considering simple harmonic oscillations and uniform applied normal edge stresses, substitution of equation (4) in equation (2) yields for each combination of m and n,

$$\begin{bmatrix} d_1 + d_2L^2 & d_3 & d_4L \\ d_3 & d_5 + d_6L^2 & d_7L \\ -d_4L & -d_7L & d_8 + d_9L^2 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ \chi \end{bmatrix} = \{0\} \tag{5}$$

where,

L denotes the operator $\dfrac{\mathrm{d}}{\mathrm{d}z}$,

$d_1 = E\lambda^2 - E_xM^2 - G_{xy}N^2,\qquad d_2 = G_{xz},\qquad d_3 = -(E_{xy}+G_{xy})MN,$

$d_4 = (E_{xz}+G_{xz})M,\qquad d_5 = E\lambda^2 - E_yN^2 - G_{xy}M^2,\qquad d_6 = G_{yz},$

$d_7 = (E_{yz}+G_{yz})N,\qquad d_8 = E\lambda^2 - G_{xz}M^2 - G_{yz}N^2,\qquad d_9 = E_z;$

$\lambda^2 = 0,$ if the plate is static and also free of body forces (6a)

$= \Omega^2\rho h^2/E,$ for simple harmonic oscillations (6b)

$= (P_xM^2 + P_yN^2)/E,$ for uniform applied normal edge stresses, P_x on $x = 0, a$ and P_y on $y = 0, b$ (6c)

For non-trivial solution of the homogeneous equation (5), the determinant of the (3×3) matrix on its left hand side must be zero. This yields

$$
\begin{aligned}
& d_2\, d_6\, d_9 (L^2)^3 + (d_6\, d_4^2 + d_2\, d_7^2 + d_2\, d_6\, d_8 + d_1\, d_6\, d_9 + d_2\, d_5\, d_9)(L^2)^2 \\
& \quad + (d_5\, d_4^2 - 2\, d_3\, d_4\, d_7 + d_1\, d_7^2 + d_1\, d_6\, d_8 + d_2\, d_5\, d_8 + d_1\, d_5\, d_9 - d_3^2\, d_9) L^2 \\
& \quad + d_8 (d_1\, d_5 - d_3^2) = 0
\end{aligned} \tag{7}
$$

for each combination of m and n. The six roots of this equation define the six values of L and correspondingly there are six arbitrary constants $A^{(i)}$ $(i = 1, 2, \ldots, 6)$. Each arbitrary constant $A^{(i)}$ is associated with an eigenvector $\{\phi^{(i)}\psi^{(i)}\chi^{(i)}\}$. The eigenvectors depend on the multiplicity of the roots and a procedure for finding them is discussed in [13]. The eigenvector corresponding to a non-repeating root $L = c$ is

$$
\begin{bmatrix} d_4 c(d_5 + d_6 c^2) - c\, d_3\, d_7 \\ d_7 c(d_1 + d_2 c^2) - c\, d_3\, d_4 \\ -(d_1 + d_2 c^2)(d_5 + d_6 c^2) + d_3^2 \end{bmatrix} A e^{cZ}
$$

where A is an arbitrary constant. The sum of the six eigenvectors corresponding to the six roots gives the total expression for ϕ, ψ and χ:

$$
\begin{bmatrix} \phi \\ \psi \\ \chi \end{bmatrix} = \sum_{i=1}^{6} \begin{bmatrix} \phi^{(i)} \\ \psi^{(i)} \\ \chi^{(i)} \end{bmatrix} A^{(i)} \tag{8}
$$

Using the stress–strain relationships (given in the notation) the following expression for stresses are obtained.

$$
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{bmatrix} \begin{bmatrix} -E_x M\phi - E_{xy} N\psi + E_{xz} L\chi \\ -E_{xy} M\phi - E_y N\psi + E_{yz} L\chi \\ -E_{xz} M\phi - E_{yz} N\psi + E_z L\chi \end{bmatrix} \sin m\pi X \sin n\pi Y \\ G_{xy}(N\phi + M\psi) \cos m\pi X \cos n\pi Y \\ G_{xz}(M\chi + L\phi) \cos m\pi X \sin n\pi Y \\ G_{yz}(N\chi + L\psi) \sin m\pi X \cos n\pi Y \end{bmatrix} \tag{9}
$$

The set of six arbitrary constants for each (m, n) in ϕ, ψ and χ is used to satisfy the six lateral surface conditions.

As mentioned earlier, extension of this analysis from homogeneous plates to laminates is straightforward. Equations (4)–(9) are valid for individual plies, the appropriate quantities for the jth ply being indicated by subscript j $(j = 1, 2, \ldots, p)$.

LATERAL SURFACE CONDITIONS

If the top and bottom surfaces are subjected to normal and shear stresses $q_{z_t}(x, y)$, $q_{xz_t}(x, y)$, $q_{yz_t}(x, y)$ and $q_{z_b}(x, y)$, $q_{xz_b}(x, y)$, $q_{yz_b}(x, y)$ respectively, (they can be either static loads or amplitudes of forcing functions with frequency Ω), then the lateral surface conditions to be satisfied are:

for homogeneous plates;

$$\begin{array}{llll} \text{at} \quad Z = 0, & \sigma_z = q_{z_t}(x, y), & \tau_{xz} = q_{xz_t}(x, y), & \tau_{yz} = q_{yz_t}(x, y) \\ \text{at} \quad Z = 1, & \sigma_z = q_{z_b}(x, y), & \tau_{xz} = q_{xz_b}(x, y), & \tau_{yz} = q_{yz_b}(x, y), \end{array} \tag{10}$$

for laminates;

$$\begin{array}{llll} \text{at} \quad Z = 0, & \sigma_{z_1} = q_{z_t}(x, y), & \tau_{xz_1} = q_{xz_t}(x, y), & \tau_{yz_1} = q_{yz_t}(x, y), \\ \text{at} \quad Z = 1, & \sigma_{z_p} = q_{z_b}(x, y), & \tau_{xz_p} = q_{xz_b}(x, y), & \tau_{yz_p} = q_{yz_b}(x, y), \end{array}$$

and at interfaces $Z = H_j (j = 1, 2, \ldots, p-1)$,

equilibrium conditions

$$\sigma_{z_j} - \sigma_{z_{j+1}} = \tau_{xz_j} - \tau_{xz_{j+1}} = \tau_{yz_j} - \tau_{yz_{j+1}} = 0$$

and continuity conditions

$$u_j - u_{j+1} = v_j - v_{j+1} = w_j - w_{j+1} = 0 \tag{11}$$

The loadings are best expressed in double Fourier series,

$$\begin{bmatrix} q_{z_t} \\ q_{xz_t} \\ q_{yz_t} \\ q_{z_b} \\ q_{xz_b} \\ q_{yz_b} \end{bmatrix} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{bmatrix} Q_{z_{t,mn}} \sin m\pi X \sin n\pi Y \\ Q_{xz_{t,mn}} \cos m\pi X \sin n\pi Y \\ Q_{yz_{t,mn}} \sin m\pi X \cos n\pi Y \\ Q_{z_{b,mn}} \sin m\pi X \sin n\pi Y \\ Q_{xz_{b,mn}} \cos m\pi X \sin n\pi Y \\ Q_{yz_{b,mn}} \sin m\pi X \cos n\pi Y \end{bmatrix} \tag{12}$$

Then, satisfaction of the lateral surface conditions, equations (10) or (11), leads to a set of $6p$ simultaneous equations for each combination of m and n. These can be put in the following form.

(a) For homogeneous plates;

$$\begin{bmatrix} [R(0)] \\ [R(1)] \end{bmatrix} \{A\} = \begin{bmatrix} Q_{z_{t,mn}} \\ Q_{xz_{t,mn}} \\ Q_{yz_{t,mn}} \\ Q_{z_{b,mn}} \\ Q_{xz_{b,mn}} \\ Q_{yz_{b,mn}} \end{bmatrix} \tag{13}$$

(b) For three-ply laminates;

$$\begin{bmatrix} [R(0)]_1 & [0] & [0] \\ [R(H_1)]_1 & -[R(H_1)]_2 & [0] \\ [S(H_1)]_1 & -[S(H_1)]_2 & [0] \\ [0] & -[R(H_2)]_2 & [R(H_2)]_3 \\ [0] & -[S(H_2)]_2 & [S(H_2)]_3 \\ [0] & [0] & [R(1)]_3 \end{bmatrix} \begin{bmatrix} \{A\}_1 \\ \{A\}_2 \\ \{A\}_3 \end{bmatrix} = \begin{bmatrix} Q_{z_{t,mn}} \\ Q_{xz_{t,mn}} \\ Q_{yz_{t,mn}} \\ 0 \\ 0 \\ \vdots \\ 0 \\ Q_{z_{b,mn}} \\ Q_{xz_{b,mn}} \\ Q_{yz_{b,mn}} \end{bmatrix} \tag{14}$$

(c) For multi-ply laminates;

Equation (14) is extended in a straightforward manner and symbolically,

$$[C]\{B\} = \{Q\} \tag{14a}$$

where $[C]$ is a square matrix made up of $[R]$, $[S]$ and null matrices, $\{B\}$ is a column matrix consisting of $\{A\}_j$'s, and $\{Q\}$ is the load matrix.

In the foregoing equations,

$$\{A\}_j = \{A^{(1)}A^{(2)} \quad \ldots. \quad A^{(6)}\}_j,$$

$$[R(Z)]_j = \begin{bmatrix} I^{(1)} & I^{(2)} & \ldots & I^{(6)} \\ J^{(1)} & J^{(2)} & \ldots & J^{(6)} \\ K^{(1)} & K^{(2)} & \ldots & K^{(6)} \end{bmatrix}_j$$

$$I^{(i)} = (-E_{xz}M\phi - E_{yz}N\psi + E_z L\chi)^{(i)}$$

$$J^{(i)} = (G_{xz}[M\chi + L\phi])^{(i)}$$

$$K^{(i)} = (G_{yz}[N\chi + L\psi])^{(i)},$$

$$[S(Z)]_j = \begin{bmatrix} \phi^{(1)} & \phi^{(2)} & \ldots & \phi^{(6)} \\ \psi^{(1)} & \psi^{(2)} & \ldots & \psi^{(6)} \\ \chi^{(1)} & \chi^{(2)} & \ldots & \chi^{(6)} \end{bmatrix}_j$$

and $[0]$ is a (3×6) null matrix. In the above equations, superscripts denote the corresponding eigenvectors and subscripts the corresponding plies. It is simple to modify equation (13) or (14) to include any displacement conditions on the exterior lateral surfaces.

In non-homogeneous problems, i.e. in those problems for which the right hand side of equation (13) or (14) is not zero, results can be evaluated by summing infinite series whose individual terms are known explicitly by solution of equation (13) or (14), to the desired accuracy by retaining sufficient number of terms. In case of homogeneous problems the determinant (order $6p$) of the square matrix on the left hand side of equation (13) or (14) must be zero and this is the characteristic equation for the corresponding (m, n) combination.

PLATE UNDER UNIFORM NORMAL STATIC LOAD ON TOP SURFACE

Consider a plate loaded only on the top surface by an uniformly distributed normal static load q_0 so that,

$$
\begin{aligned}
Q_{z_{t,mn}} &= \frac{16q_0}{mn\pi^2}, \; m \text{ and } n \text{ odd} \\
&= 0, \; m \text{ or } n \text{ even} \\
Q_{xz_{t,mn}} &= Q_{yz_{t,mn}} = Q_{z_{b,mn}} = Q_{xz_{b,mn}} = Q_{yz_{b,mn}} = 0
\end{aligned}
\tag{15}
$$

The scheme for computation is:

(i) Determine the roots of equation (7) for each ply, for each (m, n).

(ii) Find the eigenvectors, taking into account any multiplicity of roots.

(iii) Solve equations (13) or (14) for each (m, n) to determine the arbitrary constants.

(iv) Sum the series (4) and (9) for the displacements and stresses, retaining sufficient terms to achieve the desired degree of accuracy.

Numerical results are presented in Tables 2–4 for a homogeneous plate and a three-ply laminate whose material properties are indicated in Table 1. In Table 2, exact, Reissner and thin plate [12] values of $\sigma_{x,\max}$, $\sigma_{y,\max}$, $\tau_{xz,\max}$ and $w_{\max}$ are given for homogeneous plates with $a/b = 0{\cdot}5$, 1 and 2, $h/a = 0{\cdot}05$, $0{\cdot}1$ and $0{\cdot}14$. In Table 3, the σ_x, σ_y, τ_{xz} and w distributions across the thickness for a 14 per cent thick homogeneous square plate are given. In Table 4, effects of modular ratio between plies (β) are presented for three-ply square laminates with identical top and bottom plies.

The following observations can be made from the data in Tables 2–4.

(i) Thin plate theory underestimates maximum deflection. The errors in both maximum stresses and maximum deflections predicted by thin plate theory, increase as plate thickness increases. Deflections are more inaccurate than stresses.

(ii) Even for quite thick plates Reissner's theory predicts the deflections almost exactly, but for stresses it does not appear to be distinctly superior to thin plate theory.

(iii) The thin plate and Reissner's direct stress distributions across thickness (Table 3) are only slightly different from the true distributions.

(iv) Modular ratio between plies has significant effect on the errors in thin plate theory for laminates. Mostly errors increase with increasing moduli of outer plies.

It was also observed that the number of terms needed to maintain a given level of accuracy increases as plate thickness increases. To maintain $0{\cdot}1$ per cent accuracy, homogeneous square plates with $h/a = 0{\cdot}05$, $0{\cdot}1$ and $0{\cdot}14$ require retention of terms up to $m(=n) = 25$, 87 and 167 respectively. That is, the significance of the higher harmonics of loading increases with increasing thickness. Finally, from Table 3 one confirms that the mid-surface is not really a neutral surface.

FREE VIBRATION

In this case the exterior lateral surfaces are stress free and therefore,

$$Q_{z_{t,mn}} = Q_{xz_{t,mn}} = Q_{yz_{t,mn}} = Q_{z_{b,mn}} = Q_{xz_{b,mn}} = Q_{yz_{b,mn}} = 0 \tag{16}$$

TABLE 2. DEFLECTIONS AND STRESSES IN HOMOGENEOUS

h/a	$E_x w/hq_0$				σ_z/q_0			
	Present exact analysis	Reissner's theory	Thin plate theory	% error in thin plate values	Present exact analysis	Reissner's theory	Thin plate theory	% error in thin plate values
								a/b =
0·05	−21,542	−21,542	−21,201	−1·58	262·67	262·07	262·26	−0·16
0·10	−1408·5	−1408·4	−1325·1	−5·92	65·975	65·379	65·564	−0·62
0·14	−387·23	−387·27	−344·93	−10·92	33·862	33·265	33·451	−1·21
								a/b =
0·05	−10,443	−10,442	−10,246	−1·89	144·31	143·87	144·39	0·06
0·10	−688·57	−688·37	−640·39	−7·00	36·021	35·578	36·098	0·21
0·14	−191·07	−191·02	−166·70	−12.75	18·346	17·906	18·417	0·39
								a/b =
0·05	−2048·7	−2047·9	−1988·1	−2·96	40·657	40·477	40·860	0·50
0·10	−139·08	−138·93	−124·26	−10·66	10·025	9·8460	10·215	1·90
0·14	−39·790	−39·753	−32·345	−18·71	5·0364	4·8603	5·2118	3·48

q_0: normal stress on top surface ($z = 0$); w: deflection of central point ($X = Y = Z = 0{\cdot}5$); σ_x and σ_y: normal stresses at centre of top surface ($X = Y = 0{\cdot}5$, $Z = 0$); τ_{xz}: shear stress at centre of an edge ($X = 0$, $Y = Z = 0{\cdot}5$); elastic moduli are as per Table 1.

Substitution of equation (16) in equation (13) [or (14)] makes the latter homogeneous and for non-trivial solution of the problem, the determinant of the square matrix ($6p \times 6p$) on the left hand side must be zero. For each (m, n), simultaneous solution of this characteristic equation along with equation (7) yield an infinite number of frequencies, each corresponding to a different thickness mode.

Typical numerical results are presented in Tables 5–8. In Table 5 the first 8 exact eigenvalues along with the 3 by Mindlin's theory ($\kappa^2 = 5/6$) and one by thin plate analysis are given for $mh/a = 0{\cdot}1, 0{\cdot}2, \ldots, 0{\cdot}5$ and $nh/b = 0{\cdot}1, 0{\cdot}2, \ldots, 0{\cdot}5$. In Table 6, the stress and displacement distributions across thickness for a homogeneous plate are given for $mh/a = nh/b = 0{\cdot}3$. Table 7 gives the relative magnitudes of maximum stresses and displacements for antisymmetric thickness modes. In Table 8 the flexural mode eigenvalues for various β's are tabulated for three-ply laminates with identical top and bottom plies. From the data obtained the following observations are made.

(i) It is confirmed that thin plate theory frequencies are higher than true values. The errors increase with increasing mh/a or nh/b.

(ii) For homogeneous plates, all the 3 eigenvalues given by Mindlin's theory are close to the corresponding exact values.

(iii) Thin plate and Mindlin's stress and displacement distributions across thickness are significantly different from the true distributions. For the flexural modes the deviations increase with mh/a or nh/b.

(iv) Complexity of the stress and displacement distributions across thickness increases for higher order thickness modes.

ORTHOTROPIC PLATES UNDER UNIFORM SURFACE LOAD

σ_y/q_0				τ_{xz}/q_0			
Present exact analysis	Reissner's theory	Thin plate theory	% error in thin plate values	Present exact analysis	Reissner's theory	Thin plate theory	% error in thin plate values
0.5							
79.545	79.337	79.121	−0.53	−14.048	14.114	−14.154	0.75
20.204	20.001	19.780	−2.10	−6.9266	−7.0611	−7.1328	2.98
10.515	10.312	10.092	−4.02	−4.8782	−5.0445	−5.1050	4.65
1.0							
87.080	86.921	86.487	−0.68	−10.873	−10.864	−10.972	0.91
22.210	22.048	21.622	−2.65	−5.3411	−5.4267	−5.5642	4.18
11.615	11.453	11.031	−5.03	−3.7313	−3.8741	−3.9860	6.83
2.0							
54.279	54.134	53.838	−0.81	−6.2434	−6.2191	−6.4175	2.79
13.888	13.743	13.460	−3.08	−2.9573	−3.0524	−3.2402	9.57
7.2794	7.1358	6.8671	−5.66	−1.9987	−2.1428	−2.3175	15.95

(v) As the order of the thickness modes increases, the influences of transverse stresses increase.

(vi) Modular ratio between plies has significant effect on the error in flexural frequency due to thin plate assumptions and this error increases with increasing moduli of outer plies. Density ratios have only slight effect for small m and n.

TABLE 3. VARIATIONS ACROSS THICKNESS IN HOMOGENEOUS ORTHOTROPIC PLATES UNDER UNIFORM NORMAL LOAD ON TOP SURFACE

	$w(Z)/w(0.5)$		$\sigma_x(Z)/\sigma_x(0)$		$\sigma_y(Z)/\sigma_y(0)$		$\tau_{xz}(Z)/\tau_{xz}(0.5)$	
Z	True distribution	Thin plate theory	True distribution	Thin plate and Reissner theories	True distribution	Thin plate and Reissner theories	True distribution	Thin plate and Reissner theories
0	1.0000	1.0	1.0	1.0	1.0	1.0	0	0
0.1	1.0005	1.0	0.7849	0.8	0.7914	0.8	0.4992	0.36
0.2	1.0007	1.0	0.5801	0.6	0.5888	0.6	0.7325	0.64
0.3	1.0007	1.0	0.3832	0.4	0.3906	0.4	0.8906	0.84
0.4	1.0004	1.0	0.1914	0.2	0.1954	0.2	0.9789	0.96
0.5	1.0	1.0	−0.0023	0	−0.0016	0	1.0	1.0
0.6	0.9994	1.0	−0.1869	−0.2	−0.1921	−0.2	0.9540	0.96
0.7	0.9986	1.0	−0.3786	−0.4	−0.3874	−0.4	0.8388	0.84
0.8	0.9976	1.0	−0.5755	−0.6	−0.5856	−0.6	0.6492	0.64
0.9	0.9964	1.0	−0.7803	−0.8	−0.7882	−0.8	0.3758	0.36
1.0	0.9949	1.0	−0.9954	−1.0	−0.9968	−1.0	0	0

Variations of w, σ_x and σ_y along $X = Y = 0.5$ and τ_{xz} along $X = 0$, $Y = 0.5$. Elastic moduli are as per Table 1. $a/b = 1$, $h/a = 0.14$.

TABLE 4. DEFLECTIONS AND STRESSES IN THREE PLY

$\beta = E_{x_1}/E_{x_2}$	Present exact analysis				
	1	5	10	15	1
wE_{x_2}/h_{q_0}	−688·58	−258·97	−159·38	−121·72	−640·39
σ_x/q_0:					
Top ply at top surface	36·021	60·353	65·332	66·787	36·098
Top ply at interface	28·538	46·623	48·857	48·299	28·878
Mid ply at upper interface	28·538	9·3402	4·9030	3·2379	28·878
Mid ply at lower interface	−28·454	−9·2845	−4·8600	−3·2009	−28·878
Bottom ply at interface	−28·454	−46·426	−48·609	−48·028	−28·878
Bottom ply at bottom surface	−35·937	−60·155	−65·083	−66·513	−36·098
σ_y/q_0:					
Top ply at top surface	22·210	38·491	43·566	46·424	21·622
Top ply at interface	17·669	30·097	33·413	34·955	17·297
Mid ply at upper interface	17·669	6·1607	3·4995	2·4941	17·297
Mid ply at lower interface	−17·631	−6·0574	−3·3669	−2·3476	−17·297
Bottom ply at interface	−17·631	−30·322	−33·756	−35·353	−17·297
Bottom ply at bottom surface	−22·172	−38·715	−43·908	−46·821	−21·622
τ_{xz}/q_0					
At upper interface	−2·4029	−3·7194	−3·9285	−3·9559	−2·0031
At mid surface	−5·3411	−4·3641	−4·0959	−3·9638	−5·5642
At lower interface	−1·9826	−3·2675	−3·5154	−3·5768	−2·0031

q_0: normal stress on top surface ($z = 0$); w at centre $X = Y = Z = 0{\cdot}5$, σ_x and σ_y on $X = Y = 0{\cdot}5$ and τ_{xz} on $X = 0$, $Y = 0{\cdot}5$; elastic moduli of all plies as per Table 1. Top and bottom plies are identical. $h_1/h = 0{\cdot}1$, $h_2/h = 0{\cdot}8$, $a/b = 1$, $h/a = 0{\cdot}1$.

BUCKLING

The problem of buckling of simply supported homogeneous plates is formally identical to that of free vibrations, except that the eigenvalue λ should be interpreted according to equation (6c) instead of equation (6b), as for example in Table 5. As already stated, only the flexural mode eigenvalues are of significance in many buckling problems. The present analysis can be directly extended for laminates which are initially in a state of uniform strains ε_x, ε_y and $\sigma_z = 0$ throughout. Otherwise, a three-dimensional analysis of the initial state will be involved and also the governing differential equation will be complicated by the presence of variable terms (local stresses). The required initial state of uniform direct strains can be realised in experiments by compressing by pairs of smooth rigid edge blocks moving towards each other without rotation. This initial state is equivalent to one of direct stresses P_{x_j}, P_{y_j} uniform within and on the edge of each ply, such that the quantities $(F_{x_j}P_{x_j} + F_{xy_j}P_{y_j})$, $(F_{xy_j}P_{x_j} + F_{y_j}P_{y_j})$ are independent of j. In thin or moderately thick laminates, small deviations from this condition at the edges, can be ignored by invoking Saint-Venant's principle and the present analysis can still be applied to obtain satisfactory results.

Numerical results are presented in Fig. 2 and Tables 9 and 10 for plates and laminates loaded by stresses P_{x_j} on sides $x = 0$ and a only. As such, $(F_{x_j}P_{x_j})$ and $(F_{xy_j}P_{x_j})$ are the same for all plies. In other words, $\{F_{x_j}/F_{xy_j}\}$ are the same for all plies and $(P_{x_1}:P_{x_2}:\ldots) = (1/F_{x_1}:1/F_{x_2}:\ldots)$. The least buckling stress P_{x,crm_j} for any given plate dimensions and m is

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Thin plate theory			% error in thin plate theory			
5	10	15	1	5	10	15
−216·94	−118·77	−81·768	−7·00	−16·23	−25·48	−32·82
61·141	66·947	69·135	0·21	1·31	2·47	3·52
48·913	53·557	55·308	1·19	4·91	9·62	14·51
9·7826	5·3557	3·6872	1·19	4·74	9·23	13·88
−9·7826	−5·3557	−3·6872	1·49	5·36	10·20	15·19
−48·913	−53·557	−55·308	1·49	5·36	10·18	15·16
−61·141	−66·947	−69·135	0·45	1·64	2·86	3·94
36·622	40·099	41·410	−2·65	−4·86	−7·96	−10·80
29·297	32·079	33·128	−2·11	−2·66	−3·99	−5·23
5·8595	3·2079	2·2085	−2·11	−4·89	−8·33	−11·45
−5·8595	−3·2079	−2·2085	−1·89	−3·27	−4·72	−5·93
−29·297	−32·079	−33·128	−1·89	−3·38	−4·97	−6·29
−36·622	−40·099	−41·410	−2·48	−5·41	−8·67	−11·56
−3·3860	−3·7075	−3·8287	−16·64	−8·96	−5·63	−3·22
−4·5899	−4·3666	−4·2825	4·18	5·17	6·61	8·04
−3·3860	−3·7075	−3·8287	1·03	3·63	5·46	7·04

realised with only $n = 1$. For a given h/b, the buckling stress parameter k_x achieves a constant value for all integral values of a/b, and the corresponding $m = a/b$. That is, when a/b is integral the plate buckles into square panels. This value of k_x is also its asymptotic value.

In Fig. 2 the buckling stress parameter k_x is plotted against a/b for various h/b's. In Table 9 the asymptotic values of the buckling stress parameter k_x are given for homogeneous orthotropic plates with $h/b = 0{\cdot}05$, $0{\cdot}1$ and $0{\cdot}2$. In Table 10, the asymptotic value of k_x is given for three-ply laminates. From these data the following observations on buckling of orthotropic plates and laminates may be made.

(i) Thin plate theory gives optimistic values for the buckling stress, the errors increasing rapidly as the thickness increases.

(ii) Mindlin theory can yield accurate buckling stresses even for thick homogeneous plates.

(iii) Modular ratio between plies has significant effect on the errors in buckling stress due to thin plate type assumptions. The errors increase as moduli of outer plies are increased.

The stress and displacement distributions are similar to those of the corresponding flexural mode in vibration.

CONCLUDING REMARKS

An exact three-dimensional unified analysis has been established for the bending, vibration and buckling of simply supported, thick, orthotropic, rectangular plates and

TABLE 5. EIGENVALUES FOR FREE VIBRATIONS

$\frac{mh}{a}$	$\frac{nh}{b}$	Exact analysis					
		I-A	*I-S**	*II-S*	*II-A**	*III-A*	*III-S*
0·1	0·1	0·04742	0·21697	0·39405	1·3077	1·6530	2·2722
0·1	0·2	0·10329	0·34501	0·56242	1·3331	1·7160	2·2600
0·1	0·3	0·18881	0·49530	0·76004	1·3765	1·8115	2·2427
0·1	0·4	0·29690	0·65190	0·96901	1·4372	1·9306	2·2245
0·1	0·5	0·42124	0·81071	1·1825	1·5133	2·0664	2·2097
0·2	0·1	0·11880	0·35150	0·67278	1·4205	1·6805	2·2537
0·2	0·2	0·16942	0·43382	0·78796	1·4316	1·7509	2·2455
0·2	0·3	0·24753	0·55201	0·94433	1·4596	1·8523	2·2335
0·2	0·4	0·34755	0·68957	1·1231	1·5068	1·9749	2·2206
0·2	0·5	0·46428	0·83699	1·3142	1·5719	2·1122	2·2105
0·3	0·1	0·21804	0·50291	0·97278	1·5777	1·7334	2·2396
0·3	0·2	0·26244	0·56047	1·0573	1·5651	1·8195	2·2346
0·3	0·3	0·33200	0·65043	1·1814	1·5737	1·9289	2·2274
0·3	0·4	0·42242	0·76415	1·3321	1·6049	2·0546	2·2198
0·3	0·5	0·52956	0·89360	1·4994	1·6566	2·1923	2·2151
0·4	0·1	0·33189	0·65908	1·2795	1·7179	1·8548	2·2346
0·4	0·2	0·37066	0·70277	1·3453	1·6940	1·9447	2·2319
0·4	0·3	0·43225	0·77334	1·4463	1·6923	2·0534	2·2283
0·4	0·4	0·51342	0·86667	1·5736	1·7129	2·1763	2·2255
0·4	0·5	0·61092	0·97769	1·7191	1·7542	$2{\cdot}2262^{III\text{-}S}$	$2{\cdot}3104^{III\text{-}A}$
0·5	0·1	0·45265	0·81720	1·5890	1·8056	2·0667	2·2395
0·5	0·2	0·48680	0·85223	1·6425	1·7974	2·1344	2·2386
0·5	0·3	0·54160	0·90962	1·7266	1·7999	2·2288	2·2380
0·5	0·4	0·61465	0·98732	$1{\cdot}8187^{II\text{-}A}$	$1{\cdot}8350^{II\text{-}S}$	$2{\cdot}2394^{III\text{-}S}$	$2{\cdot}3412^{III\text{-}A}$
0·5	0·5	0·70338	1·0824	$1{\cdot}8559^{II\text{-}A}$	$1{\cdot}9596^{II\text{-}S}$	$2{\cdot}2468^{III\text{-}S}$	$2{\cdot}4665^{III\text{-}A}$

* Pure thick–twist modes.
Elastic moduli as per Table 1.
Eigenvalue $\lambda = \Omega\sqrt{\rho h^2/E_x}$ for free vibrations
$= \sqrt{P_{x_{crm}} M^2/E_x}$ for buckling under uniform normal edge stresses P_x on $x = 0$ and a.
A and S denote modes which are antisymmetric and symmetric about mid-plane respectively.

laminates. The solution is in series form and each term in it is explicitly defined by a set of simultaneous equations. Thus it is easy to sum the series to any desired degree of accuracy in non-homogeneous problems like plates under static or dynamic loads on lateral surfaces, while in homogeneous problems like free vibration or buckling from a state of uniform strain closed form characteristic equations are obtained to define natural frequencies or buckling stresses. This type of solution was made possible by the choice of $v = 0$, instead of $\tau_{xy} = 0$, as one of the "simple" support conditions along the straight edge $x = \text{const.}$ In fact equation (4) is only one of the twelve sets of series providing the general solution [14] of the governing differential equation. If the condition $\tau_{xy} = 0$ is to be satisfied the series in equation (4), should be augmented by additional series. Then the terms are no longer explicitly determinable nor is the characteristic equation in closed form. There is consequent loss of elegance and simplicity and enormous increase in computation. An example of an isotropic square plate has been worked out [14] to examine whether the two modes of defining a simple support would lead to significantly different results. The $v = 0$ condition

AND BUCKLING OF HOMOGENEOUS PLATES

		Mindlin's analysis			Thin plate theory	% error in thin plate values
IV-S*	V-S	I-A	II-A*	III-A	I-A	
2·5479	3·2636	0·04740	1·3159	1·6646	0·04967	4·74
2·5604	3·3043	0·10317	1·3411	1·7305	0·11200	8·44
2·5824	3·3678	0·18839	1·3841	1·8306	0·21537	14·07
2·6145	3·4497	0·29588	1·4445	1·9560	0·35993	21·23
2·6565	3·5458	0·41927	1·5204	2·1001	0·54574	29·56
2·6264	3·2760	0·11873	1·4285	1·6921	0·13538	13·95
2·6334	3·3179	0·16918	1·4393	1·7655	0·19866	17·26
2·6481	3·3828	0·24692	1·4671	1·8715	0·30289	22·37
2·6727	3·4661	0·34628	1·5142	2·0002	0·44802	28·91
2·7081	3·5633	0·46201	1·5792	2·1456	0·63418	36·59
2·7383	3·2967	0·21776	1·5857	1·7450	0·27789	27·45
2·7392	3·3409	0·26192	1·5727	1·8341	0·34176	30·22
2·7457	3·4085	0·33101	1·5813	1·9480	0·44699	34·64
2·7614	3·4942	0·42066	1·6127	2·0796	0·59313	40·41
2·7882	3·5933	0·52671	1·6648	2·2250	0·78011	47·31
2·8756	3·3261	0·33110	1·7266	1·8657	0·47734	43·82
2·8695	3·3743	0·36955	1·7022	1·9587	0·54152	46·09
2·8668	3·4463	0·43056	1·7006	2·0716	0·64750	49·80
2·8730	3·5356	0·51084	1·7218	2·2003	0·79466	54·78
2·8905	3·6373	0·60716	1·7638	2·3419	0·98269	60·85
3·0334	3·3648	0·45087	1·8150	2·0770	0·73374	62·10
3·0174	3·4204	0·48463	1·8067	2·1475	0·79809	63·95
3·0040	3·4991	0·53877	1·8095	2·2459	0·90459	67·02
3·0000	3·5931	0·61083	1·8292	2·3639	1·0526	71·25
3·0082	3·6979	0·69828	1·8665	2·4965	1·2416	76·52

is found to predict smaller central deflections, lower by 7·4 per cent, and 9·4 per cent for 10 per cent and 14 per cent thick plates.

The effect of thin plate assumptions is to increase the stiffness of the structure and therefore yield lower deflections, higher flexural frequencies and optimistic buckling stresses. The errors increase as the thickness increases, and in vibrations, also with increasing order of the mode as m or n increases. In laminates the modular ratios between plies have very significant effects on the errors due to thin plate type of assumptions and these errors increase with increasing moduli of outer plies. In vibrations thin plate theory yields only one doubly infinite spectrum of frequencies.

In orthotropic plates, the errors due to thin plate assumptions depend strongly also on the extent of orthotropy, i.e. the relative values of various elastic moduli. The errors in thin plate theory increase as the relative magnitudes of transverse elastic flexibility constants ($F_z, F_{xz}, F_{yz}, 1/G_{xz}, 1/G_{yz}$) increase relative to inplane constants ($F_x, F_y, F_{xy}, 1/G_{xy}$). In isotropic plates, it was possible to define a "thin plate". For example tolerating 1 per cent inaccuracy, a plate could be called "thin" if it is under 5 per cent thick when subjected to uniform loading or when $\{(mh/a)^2+(nh/b)^2\} \leq 1/250$ for free vibrations [7, 8]. Because of the large number of parameters involved, it is rather difficult to define the term "thin" for orthotropic plates.

TABLE 6. VARIATIONS OF STRESSES AND DISPLACEMENTS ACROSS THICKNESS FOR A HOMOGENEOUS ORTHOTROPIC PLATE IN FREE VIBRATION

Z	$\frac{\sigma_x}{\sigma_x(0)}$	$\frac{\sigma_y}{\sigma_y(0)}$	$\frac{\sigma_z}{\sigma_z(0\cdot4)}$	$\frac{\tau_{xy}}{\tau_{xy}(0)}$	$\frac{\tau_{xz}}{\tau_{xz}(0\cdot5)}$	$\frac{\tau_{yz}}{\tau_{yz}(0\cdot5)}$	$\frac{u}{u(0)}$	$\frac{v}{v(0)}$	$\frac{w}{w(0)}$
(a) Thin plate theory and the three thickness-modes of Mindlin's theory									
0	1·0	1·0	—	1·0	0	0	1·0	1·0	1·0
0·1	0·8	0·8	—	0·8	0·36	0·36	0·8	0·8	1·0
0·2	0·6	0·6	—	0·6	0·64	0·64	0·6	0·6	1·0
0·3	0·4	0·4	—	0·4	0·84	0·84	0·4	0·4	1·0
0·4	0·2	0·2	—	0·2	0·96	0·96	0·2	0·2	1·0
0·5	0	0	—	0	1·0	1·0	0	0	1·0
(b) Exact theory									
First antisymmetric thickness mode (I-A)									
0	1·0	1·0	0	1·0	0	0	1·0	1·0	1·0
0·1	0·7134	0·7539	1·5685	0·7360	0·3943	0·3835	0·6963	0·7644	1·0054
0·2	0·4871	0·5396	2·0482	0·5163	0·6738	0·6632	0·4650	0·5532	1·0083
0·3	0·3028	0·3479	1·7657	0·3278	0·8595	0·8534	0·2838	0·3595	1·0096
0·4	0·1450	0·1704	1·0	0·1591	0·9655	0·9638	0·1343	0·1770	1·0102
0·5	0	0	0	0	1·0	1·0	0	0	1·0103
Second antisymmetric thickness mode (II-A)									
0	1.0	1·0	0	1·0	0	0	1·0	1·0	1·0
0·1	0·9511	0·9509	1·3346	0·9512	0·3090	0·3090	0·9511	0·9510	16·658
0·2	0·8090	0·8088	1·8798	0·8092	0·5878	0·5878	0·8090	0·8089	30·755
0·3	0·5878	0·5875	1·7099	0·5880	0·8090	0·8090	0·5878	0·5877	41·927
0·4	0·3090	0·3089	1·0	0·3091	0·9511	0·9511	0·3090	0·3090	49·095
0·5	0	0	0	0	1·0	1·0	0	0	51·563
Third antisymmetric thickness mode (III-A)									
0	1·0	1·0	0	1·0	0	0	1·0	1·0	1·0
0·1	0·9837	0·9516	1·3098	0·9789	0·2942	0·3021	0·9922	0·9721	0·9023
0·2	0·8573	0·8111	1·8614	0·8501	0·5719	0·5804	0·8698	0·8399	0·7710
0·3	0·6333	0·5905	1·7037	0·6263	0·7992	0·8045	0·6450	0·6167	0·6439
0·4	0·3361	0·3109	1·0	0·3320	0·9482	0·9497	0·3431	0·3262	0·5533
0·5	0	0	0	0	1·0	1·0	0	0	0·5206
First symmetric thickness mode (I-S)					$\frac{\tau_{xz}}{\tau_{xz}(0\cdot2)}$	$\frac{\tau_{yz}}{\tau_{yz}(0\cdot2)}$			
0	1·0	1·0	0	1·0	0	0	1·0	1·0	1·0
0·1	0·9790	1·0130	0·3868	1·0415	0·7609	0·7614	0·9890	1·0074	0·8112
0·2	0·9636	1·0228	0·6777	1·0723	1·0	1·0	0·9811	1·0131	0·6148
0·3	0·9533	1·0297	0·8804	1·0925	0·8661	0·8656	0·9757	1·0170	0·4129
0·4	0·9472	1·0337	1·0	1·1060	0·4919	0·4915	0·9727	1·0194	0·2073
0·5	0·9453	1·0350	1·0395	1·1101	0	0	0·9717	1·0202	0
Second symmetric thickness mode (II-S)									
0	1·0	1·0	0	1·0	0	0	1·0	1·0	1·0
0·1	1·0077	1·0062	0·3602	1·0091	0·7241	0·7336	1·0073	1·0120	0·8362
0·2	1·0139	1·0114	0·6526	1·0165	1·0	1·0	1·0131	1·0221	0·6489
0·3	1·0184	1·0152	0·8680	1·0220	0·8968	0·8887	1·0173	1·0296	0·4431
0·4	1·0211	1·0176	1·0	1·0254	0·5197	0·5122	1·0199	1·0342	0·2248
0·5	1·0221	1·0184	1·0444	1·0265	0	0	1·0208	1·0358	0

Elastic moduli as per Table 1; $mh/a = nh/b = 0{\cdot}3$.

The improved approximate theories like Reissner's and Mindlin's are good even for quite thick plates for specific purposes. For example, Reissner's theory predicts deflections accurately, while Mindlin's theory yields accuracte frequencies for the first 3 antisymmetric thickness modes of free vibrations.

TABLE 7. RELATIVE MAGNITUDES OF MAXIMUM STRESSES AND DISPLACEMENTS FOR ANTISYMMETRIC THICKNESS MODES OF HOMOGENEOUS ORTHOTROPIC PLATES IN FREE VIBRATION

	$\frac{\sigma_{y,max}}{\sigma_{x,max}}$	$\frac{\sigma_{z,max}}{\sigma_{x,max}}$	$\frac{\tau_{xy,max}}{\sigma_{x,max}}$	$\frac{\tau_{xz,max}}{\sigma_{x,max}}$	$\frac{\tau_{yz,max}}{\sigma_{x,max}}$	$\frac{u_{max}}{w_{max}}$	$\frac{v_{max}}{w_{max}}$
Thin plate theory (I-*A*)	0·6142	0	0·4272	0·2932	0·2023	0·4712	0·4712
Mindlin's theory							
Thickness mode I-*A*	0·7693	0	0·4937	0·3306	0·2644	0·2290	0·3550
Thickness mode II-*A*	0·0174	0	0·1563	0·4716	0·3906	40·27	18·93
Thickness mode III-*A*	0·4804	0	0·3698	1·032	0·5694	6·908	4·160
Exact theory							
Thickness mode I-*A*	0·7278	0·0301	0·4759	0·2827	0·2290	0·2677	0·3727
Thickness mode II-*A*	0·0565	0·0002	0·1396	0·6128	0·5355	41·62	22·26
Thickness mode III-*A*	0·8624	0·1657	0·5336	0·4407	1·310	1·898	3·680

$mh/a = nh/b = 0{\cdot}3$.

TABLE 8. FLEXURAL MODE EIGENVALUES FOR FREE VIBRATIONS OF THREE-PLY ORTHOTROPIC LAMINATES

Density ratio ρ_1/ρ_2	$\beta = E_{x_1}/E_{x_2}$	Exact value	Thin plate value	% error in thin plate value
1	1	0·047419	0·049666	4·74
1	2	0·057041	0·060584	6·21
1	5	0·077148	0·085333	10·61
1	10	0·098104	0·115328	17·56
1	15	0·112034	0·138994	24·06
3	15	0·094548	0·117471	24·25

Elastic moduli of all plies as per Table 1. Top and bottom plies are identical. $h_1/h = 0{\cdot}1$, $h_2/h = 0{\cdot}8$, $mh/a = nh/b = 0{\cdot}1$. Eigenvalue $= \Omega\sqrt{\rho_2 h^2/E_{x_2}}$.

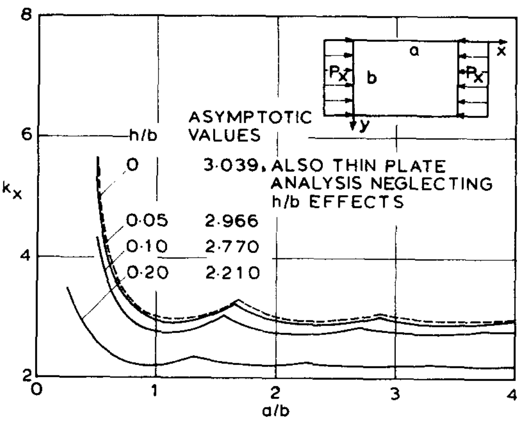

FIG. 2. Variation of buckling stress parameter k_x with h/b and a/b for plate loaded on $x = 0, a$. $k_x = (P_{x_{crm}}/E)(12/\pi^2)(b/h)^2$.

TABLE 9. ASYMPTOTIC k_x FOR BUCKLING OF HOMOGENEOUS ORTHOTROPIC PLATES UNDER UNIFORM NORMAL STRESS P_x ON EDGES $x = 0$ AND a

h/b	Exact theory	Mindlin's theory	Thin plate theory	% error in thin plate values
0·05	2·966	2·965	3·039	2·46
0·1	2·770	2·768	3·039	9·71
0·2	2·210	2·204	3·039	37·5

Elastic moduli as per Table 1. $k_x = \frac{12}{\pi^2}\frac{P_{xcrm}}{E_x}\left(\frac{b}{h}\right)^2$.

TABLE 10. ASYMPTOTIC k_x FOR BUCKLING OF THREE-PLY ORTHOTROPIC LAMINATES FROM AN INITIAL STATE OF UNIFORM STRAIN

β	Exact value	Thin plate value	% error in thin plate value
1	2·770	3·039	9·7
2	3·330	3·768	13·2
5	4·046	4·984	23·2
10	4·200	5·852	39·3
15	4·037	6·263	55.1

Elastic moduli of all plies are as per Table 1. Top and bottom plies are identical.

$$\beta = E_{x_1}/E_{x_2} = E_{x_3}/E_{x_2}$$

$$\mu = P_{x,crmi}/E_{x_i} \quad (i = 1, 2, 3)$$

$$k_x = 12\mu(b/h)^2/\pi^2$$

$$h_1/h = 0{\cdot}1, \quad h_2/h = 0{\cdot}8, \quad h/b = 0{\cdot}1.$$

Thus, a three-dimensional elasticity solution is necessary when both stresses and displacements are required in non-homogeneous problems or for establishing the full spectrum of modes in free vibrations. The analysis presented here is for general orthotropic laminates with arbitrary properties for each ply, except in buckling when the relative elastic moduli must be identical for all the plies. Analysis of sandwich plates with core having only shear rigidity is a special case of the above analysis. Tetragonal, cubic and isotropic materials are special cases of orthotropic materials and the present analysis is applicable.

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Абстракт—Дается обьединенный точный анализ статики и динамики некоторого класса толстых слоистых пластиков. Выводится трехмерная, линейная теория малых деформаций, в рамках теории упругости для изгиба, колебаний и потери устойчивости, свободно опертых толстых ортотропных прямоугольных пластин и слоистых пластиков. Решение является формально точным и приводит к несложным бесконечным рядам для напряжений и перемещений при изгибе, вынужденным колебаниям и задачам типа "балка-колонна". Для свободных же колебаний и задач устойчивости решение приводит к характеристическим уравнениям в замкнутом виде. В случае свободных колебаний пластинок, предлагаемый анализ дает тройный бесконечный спектр для частот, вместе только двойного бесконечного спектра, в рамках теории тонких пластинок или трех двойных. бесконечных спектров при анализе типа Рейсснера и Миндлина. Для пластин и слоистых пластиков даются некоторые численные результаты. Сравнение результатов теории тонких пластинок и теорией Рейсснера и Миндлина с результатами предложенными в работе дает некотоые важнрые заключения относительно важности и эффектов предположений, сделанных в приближенных теориях.